Exam Introduction to Mathematics Part 1:

Mathematical Modelling - Dimensional Analysis

September 25th, 2017: 9.00-10.00.

This exam has 2 problems. Each problem is worth 5 points; more details can be found below. Write on each page your name and student number, and on the first page your seminar group. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by work. Success.

1. A ball is released under water. It moves with velocity v to the surface. The rise velocity v depends on the mass density ρ_b and the radius R of the ball, the gravitational constant g, and the mass density ρ_w and kinematic viscosity ν of water. The dimension of ν is given by $[\nu] = L^2/T$, where L and T denote length and time, respectively

   (a) (*4 points*) Apply a dimensional analysis to find a dimensionally reduced form for v.

   (b) (*1 point*) Stokes law states that

   $$v = \frac{2gR^2(\rho_b - \rho_w)}{9\nu\rho_w}$$

   How does this differ from your result in (a)?

2. The concentration $c(x,t)$ of a chemical over a interval $0 < x < \ell$ satisfies the weakly nonlinear diffusion equation

   $$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + \lambda(\gamma - c)c$$

   where the boundary conditions are $c(x=0,t) = c(x=\ell,t) = 0$ for all $t > 0$, and the initial condition is $c(x,t=0) = c_0 \sin(2\pi x/\ell)$. The above partial differential equation is known as Fisher's equation, and it arises in the study of the movement of genetic traits in a population. A common simplifying assumption made when studing this equation is that the nonlinearity is weak, which means that the term $\lambda(\gamma - c)c$ is small in comparison to the others in the differential equation. This assumption is to be accounted for in the nondimensionalization.

   (a) (*1.5 point*) Before starting the nondimensionalization process the fundamental dimensions of the variables and parameters in the problem are to be determined. The concentration c corresponds to the number of molecules per unit volume, and so $[c] = L^{-3}$. Determine the fundamental dimensions of the other variables and parameters.

   (b) (*3.5 points*) To nondimensionalize the problem we introduce the change of variables

   $$x = x_c\, y \qquad t = t_c\, s \qquad c = c_c\, u$$

   where x_c has the dimensions of length and is a characteristic value of x. Similar, t_c and c_c have dimension time and concentration, respectively. Apply Holmes' rules of thumb to nondimensionalize the problem. That is, determine x_c, t_c and c_c such that the initial and boundary conditions have no nondimensional groups and the reduced problem (no nonlinear diffusion) contains no nondimensional group too.